Online Supplement to

"A Social Interaction Model with Ordered Choices"

Xiaodong Liu and Jiannan Zhou

Department of Economics, University of Colorado, Boulder, CO 80309, U.S.A.

September 20, 2017

A Proofs

Proof of Proposition 1. Under Assumption 1 (i), $\vec{h}(\cdot)$ is continuous, and thus the existence of a solution to the system of equations

$$\mathbf{y}^{\mathrm{E}} = \vec{h}(\mathbf{y}^{\mathrm{E}}), \tag{A.1}$$

is guaranteed by the Brouwer fixed-point theorem. By the contraction mapping theorem, the system of equations (A.1) has a unique solution if $\vec{h}(\cdot)$ is a contraction mapping with respect to some matrix norm $||\cdot||$. Thus, to show the desired result, we only need to show that, under Assumption 1 (ii), $\vec{h}(\cdot)$ is a contraction mapping.

First, we show that $\vec{h}(\cdot)$ is a contraction mapping with respect to a (submultiplicative) matrix norm $||\cdot||$, if $||\vec{h}'(\cdot)|| < 1$ where $\vec{h}'(\mathbf{x}) = \partial\vec{h}(\mathbf{x})/\partial\mathbf{x}'$. Let $\vec{k}(t) = \vec{h}(\mathbf{x}_1 + t(\mathbf{x}_2 - \mathbf{x}_1))$. If $||\vec{h}'(\cdot)|| < 1$, then

$$\begin{aligned} ||\vec{h}(\mathbf{x}_2) - \vec{h}(\mathbf{x}_1)|| &= ||\vec{k}(1) - \vec{k}(0)|| = \left\| \int_0^1 \vec{k}'(t)dt \right\| = \left\| \int_0^1 (\mathbf{x}_2 - \mathbf{x}_1)\vec{h}'(\mathbf{x}_1 + t(\mathbf{x}_2 - \mathbf{x}_1))dt \right\| \\ &\leq \int_0^1 ||\mathbf{x}_2 - \mathbf{x}_1|| \cdot ||\vec{h}'(\mathbf{x}_1 + t(\mathbf{x}_2 - \mathbf{x}_1))||dt < ||\mathbf{x}_2 - \mathbf{x}_1||, \end{aligned}$$

i.e. $\vec{h}(\cdot)$ is a contraction mapping.

Next, as the (i,j)-th element of $\partial\vec{h}(\mathbf{y}^{\mathrm{E}})/\partial\mathbf{y}^{\mathrm{E}\prime}$ is given by

$$\partial h_i(\mathbf{y}^{\mathrm{E}})/\partial y_j^{\mathrm{E}} = \lambda w_{ij} \textstyle\sum_{k=1}^{m-1} f(\alpha_{m-k} - \lambda \sum_{j=1}^{n} w_{ij} y_j^{\mathrm{E}} - \mathbf{x}_i'\boldsymbol{\beta}),$$

we have

$$|\partial h_i(\mathbf{y}^{\mathrm{E}})/\partial y_j^{\mathrm{E}}| \leq (m-1)|\lambda| \cdot |w_{ij}| \cdot \sup_u f(u).$$

Hence, under Assumption 1 (ii), either

$$||\partial\vec{h}(\mathbf{y}^{\mathrm{E}})/\partial\mathbf{y}^{\mathrm{E}\prime}||_\infty \leq (m-1)|\lambda| \cdot ||\mathbf{W}||_\infty \sup_u f(u) < 1,$$

or

$$||\partial\vec{h}(\mathbf{y}^{\mathrm{E}})/\partial\mathbf{y}^{\mathrm{E}\prime}||_1 \leq (m-1)|\lambda| \cdot ||\mathbf{W}||_1 \sup_u f(u) < 1.$$

Therefore, under Assumption 1 (ii), the system of equations (A.1) is a contraction mapping with respect to the $||\cdot||_\infty$ or $||\cdot||_1$ norm, whichever is a smaller matrix norm of $\mathbf{W}$. □

Proof of Proposition 2. Given the network topology $\mathbf{W}$, observational equivalence requires

$$\Pr(y_i \leq k|\mathbf{W},\mathbf{X}) = F(\alpha_k - \lambda\mathbf{w}_i\mathbf{y}^{\mathrm{E}} - \mathbf{x}_i'\boldsymbol{\beta}) = F(\widetilde{\alpha}_k - \widetilde{\lambda}\mathbf{w}_i\widetilde{\mathbf{y}}^{\mathrm{E}} - \mathbf{x}_i'\widetilde{\boldsymbol{\beta}}),$$

for $k = 1, \cdots, m-1$ and for any $\mathbf{X}$ in its support, where, under Assumption 1, $\mathbf{y}^{\mathrm{E}}$ and $\widetilde{\mathbf{y}}^{\mathrm{E}}$ are uniquely determined by

$$y_i^{\mathrm{E}} = m - \textstyle\sum_{k=1}^{m-1} F(\alpha_k - \lambda\mathbf{w}_i\mathbf{y}^{\mathrm{E}} - \mathbf{x}_i'\boldsymbol{\beta}) \quad \text{(A.2)}$$

$$\widetilde{y}_i^{\mathrm{E}} = m - \textstyle\sum_{k=1}^{m-1} F(\widetilde{\alpha}_k - \widetilde{\lambda}\mathbf{w}_i\widetilde{\mathbf{y}}^{\mathrm{E}} - \mathbf{x}_i'\widetilde{\boldsymbol{\beta}}) \quad \text{(A.3)}$$

for $i = 1, \cdots, n$, respectively. The model is identified if observational equivalence of $\boldsymbol{\delta}$ and $\widetilde{\boldsymbol{\delta}}$ implies $\boldsymbol{\delta} = \widetilde{\boldsymbol{\delta}}$. That is,

$$F(\alpha_k - \lambda\mathbf{w}_i\mathbf{y}^{\mathrm{E}} - \mathbf{x}_i'\boldsymbol{\beta}) = F(\widetilde{\alpha}_k - \widetilde{\lambda}\mathbf{w}_i\widetilde{\mathbf{y}}^{\mathrm{E}} - \mathbf{x}_i'\widetilde{\boldsymbol{\beta}}), \quad \text{(A.4)}$$

for $k = 1, \cdots, m-1$, implies $\boldsymbol{\delta} = \widetilde{\boldsymbol{\delta}}$. If (A.4) holds for $k = 1, \cdots, m-1$, then the right hand sides of (A.2) and (A.3) are identical, i.e., $y_i^{\mathrm{E}} = \widetilde{y}_i^{\mathrm{E}}$, for $i = 1, \cdots, n$. Therefore, (A.4) can be rewritten as

$$F(\alpha_k - \lambda \mathbf{w}_i \mathbf{y}^{\mathrm{E}} - \mathbf{x}_i' \boldsymbol{\beta}) = F(\widetilde{\alpha}_k - \widetilde{\lambda} \mathbf{w}_i \mathbf{y}^{\mathrm{E}} - \mathbf{x}_i' \widetilde{\boldsymbol{\beta}}), \tag{A.5}$$

for $k = 1, \cdots, m-1$. Under Assumption 2, (A.5) implies

$$(\alpha_k - \widetilde{\alpha}_k)\boldsymbol{\iota}_n + (\lambda - \widetilde{\lambda})\mathbf{W}\mathbf{y}^{\mathrm{E}} + \mathbf{X}(\boldsymbol{\beta} - \widetilde{\boldsymbol{\beta}}) = 0,$$

which, in turn, implies $\alpha_k = \widetilde{\alpha}_k$, $\lambda = \widetilde{\lambda}$ and $\boldsymbol{\beta} = \widetilde{\boldsymbol{\beta}}$, for $k = 1, \cdots, m-1$. Hence, observational equivalence of $\boldsymbol{\delta}$ and $\widetilde{\boldsymbol{\delta}}$ implies $\boldsymbol{\delta} = \widetilde{\boldsymbol{\delta}}$, i.e., the model is identified. □